

Twists and quantizations of Cartan type H Lie algebras

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ABSTRACT. We construct explicit Drinfel'd twists for the generalized Cartan type H Lie algebras and obtain the corresponding quantizations. By modular reduction and base changes, we obtain certain modular quantizations of the restricted universal enveloping algebra $\mathbf{u}(\mathbf{H}(n; \underline{1}))$ in characteristic p . They are new Hopf algebras of truncated p -polynomial noncommutative and non-cocommutative deformation of prime-power dimension $p^{p^{2n}-1}$, which contain the well-known Radford algebra [21] as a Hopf subalgebra. As a by-product, we also get some Jordanian quantizations for \mathfrak{sp}_{2n} .

This paper is a continuation of [13, 14] in which quantizations of Cartan type Lie algebras of types W and S were studied. In the present paper, we continue to treat the same questions both for the generalized Cartan type H Lie algebras in characteristic 0 (for the definition, see [20]) and for the restricted Hamiltonian algebra $\mathbf{H}(n; \underline{1})$ in the modular case (for the definition, see [24], [25]).

Although, in principle, the possibility to quantize an arbitrary Lie bialgebra has been proved ([6, 7, 5, 8, 9], etc.), an explicit formulation of Hopf operations remains nontrivial. In particular, only a few kinds of twists were known with explicit expressions, see [10, 15, 16, 19, 22] and the references therein. In this research, we start with an explicit Drinfel'd twist due to [10, 12] and in fact, this Drinfel'd twist is essentially a variation (see the proof in [14]) of the Jordanian twist which first appeared, using different expression, in Coll-Gerstenhaber-Giaquinto's paper [2], and recently used extensively by Kulish et al (see [15, 16], etc.). Using this explicit Drinfel'd twist we obtain *vertical basic* twists and *horizontal basic* twists for the generalized Cartan type H Lie algebras and the corresponding quantizations in characteristic 0. These basic twists can afford many more Drinfel'd twists, likewise on type S . To study the modular case, what we discuss first involves the arithmetic

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property of quantizations, for working out their quantization integral forms. To this end, we have to work over the so-called “*positive*” part subalgebra \mathbf{H}^+ of the generalized Cartan type H Lie algebra. This is the crucial observation here. It is an infinite-dimensional simple Lie algebra defined over a field of characteristic 0, while, over a field of characteristic p , it contains a maximal ideal $J_{\underline{1}}$ and the corresponding quotient is exactly the algebra $\mathbf{H}'(2n; \underline{1})$. Its derived subalgebra $\mathbf{H}(2n; \underline{1}) = \mathbf{H}'(2n; \underline{1})^{(1)}$ is a Cartan type restricted simple modular Lie algebra of H type. Secondly, in order to yield the *expected* finite-dimensional quantizations of the restricted universal enveloping algebra of the Hamiltonian algebra $\mathbf{H}(2n; \underline{1})$, we need to carry out the modular reduction process: *modulo p reduction* and *modulo “ p -restrictedness” reduction*, during which we have to take the suitable *base changes*. These are the other two crucial technical points. Our work gets a new class of noncommutative and noncocommutative Hopf algebras of prime-power dimension in characteristic p , which is significant to recognize the Kaplansky’s problem.

The paper is organized as follows. In Section 1, we recall some definitions and basic facts related to the Cartan type H Lie algebra and Drinfel’d twist. In Section 2, we construct the Drinfel’d twists for the generalized Cartan type H Lie algebra, including *vertical basic* twists and *horizontal basic* twists. In Section 3, we quantize explicitly Lie bialgebra structures of the generalized Cartan type H Lie algebra by the *vertical basic* Drinfel’d twists, and using the similar methods as in type S , we obtain the quantizations of the restricted universal enveloping algebra of the Hamiltonian algebra $\mathbf{H}(2n; \underline{1})$. In Section 4, using the *horizontal* twists, we get some new quantizations of horizontal type of $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$, which contain some quantizations of the Lie algebra \mathfrak{sp}_{2n} derived from the Jordanian twists (cf. [16]).

1. Preliminaries

1.1. The generalized Cartan type H Lie algebra and its subalgebra \mathbf{H}^+ . We recall the definition of the generalized Cartan type H Lie algebra from [20] and some basics about the structure.

Let \mathbb{F} be a field with $\text{char}(\mathbb{F}) = 0$ and $n > 0$. Let $\mathbb{Q}_{2n} = \mathbb{F}[x_{-1}^{\pm 1}, \dots, x_{-n}^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial algebra and ∂_i coincide with degree operator $x_i \frac{\partial}{\partial x_i}$. Set $T = \bigoplus_{i=1}^n (\mathbb{Z}\partial_i \oplus \mathbb{Z}\partial_{-i})$, and $x^\alpha = x_{-1}^{\alpha_{-1}} \cdots x_{-n}^{\alpha_{-n}} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_{-1}, \dots, \alpha_{-n}, \alpha_1, \dots, \alpha_n) \in \mathbb{Z}^{2n}$. In particular, $x_i = x^{\epsilon_i}$, $\epsilon_i = (\delta_{-1,i}, \dots, \delta_{-n,i}, \delta_{1,i}, \dots, \delta_{n,i})$. We can define a bilinear map

$$\begin{aligned} \langle \cdot, \cdot \rangle : T \times \mathbb{Z}^{2n} &\longrightarrow \mathbb{Z} \\ \langle \partial, \alpha \rangle &\longrightarrow \sum_{i=1}^n a_i \alpha_i + a_{-i} \alpha_{-i} \end{aligned}$$

for $\partial = \sum_{i=1}^n a_i \partial_i + a_{-i} \partial_{-i} \in T$ and $\alpha = (\alpha_{-1}, \dots, \alpha_{-n}, \alpha_1, \dots, \alpha_n) \in \mathbb{Z}^{2n}$. It is easy to see that this bilinear map is non-degenerate in the sense that:

$$\begin{aligned} \partial(\alpha) &= \langle \partial, \alpha \rangle = 0 \quad (\forall \partial \in T), \implies \alpha = 0; \\ \partial(\alpha) &= \langle \partial, \alpha \rangle = 0 \quad (\forall \alpha \in \mathbb{Z}^{2n}), \implies \partial = 0. \end{aligned}$$

Define a linear map:

$$\begin{aligned} D_H : \mathbb{Q}_{2n} &\longrightarrow \text{Der}(\mathbb{Q}_{2n}) \\ x^\alpha &\longmapsto \sum_{i=1}^n x^{\alpha - \epsilon_i - \epsilon_i} (\partial_{-i}(\alpha) \partial_i - \partial_i(\alpha) \partial_{-i}) \\ &= \sum_{i=1}^n x^{\alpha - \epsilon_i - \epsilon_i} (\alpha_{-i} \partial_i - \alpha_i \partial_{-i}). \end{aligned}$$

We can see that the kernel of this map is \mathbb{F} . The image of this map $D_H(\mathbb{Q}_{2n})$ is a Lie algebra under the bracket:

$$\begin{aligned} [D_H(x^\alpha), D_H(x^\beta)] &= D_H \left(\sum_{i=1}^n \left(\frac{\partial x^\alpha}{\partial x_{-i}} \frac{\partial x^\beta}{\partial x_i} - \frac{\partial x^\alpha}{\partial x_i} \frac{\partial x^\beta}{\partial x_{-i}} \right) \right) \\ &= D_H \left(\sum_{i=1}^n (\partial_{-i}(\alpha) \partial_i(\beta) - \partial_i(\alpha) \partial_{-i}(\beta)) x^{\alpha + \beta - \epsilon_i - \epsilon_{-i}} \right) \\ &= D_H \left(\sum_{i=1}^n (\alpha_{-i} \beta_i - \alpha_i \beta_{-i}) x^{\alpha + \beta - \epsilon_i - \epsilon_{-i}} \right). \end{aligned}$$

The derived algebra $\mathbf{H} = [D_H(\mathbb{Q}_{2n}), D_H(\mathbb{Q}_{2n})]$, is the generalized Cartan type H Lie algebra, which is known to be a simple algebra, and its basis is $\{D_H(x^\alpha) \mid \alpha \in \mathbb{Z}^{2n} \setminus \{0\}\}$ (see [20]).

Define $D_i = \frac{\partial}{\partial x_i}$. For $D_H(x^\alpha) = \sum_{i=1}^n x^{\alpha - \epsilon_i - \epsilon_{-i}} (\alpha_{-i} \partial_i - \alpha_i \partial_{-i})$, we have $D_H(x^\alpha) = \sum_{i=1}^n \alpha_{-i} x^{\alpha - \epsilon_i - \epsilon_{-i}} D_i - \alpha_i x^{\alpha - \epsilon_i - \epsilon_{-i}} D_{-i}$. Set $\mathbf{H}^+ = \text{Span}_{\mathcal{K}}\{D_H(x^\alpha) \mid \alpha \in \mathbb{N}^{2n}\}$, which via the identification $x^\alpha D_i$ with $x^{\alpha - \epsilon_i} \partial_i$ (here $\alpha - \epsilon_i \in \mathbb{Z}^{2n}$), can be considered as a Lie subalgebra (the “positive” part) of the generalized Cartan type H Lie algebra over a field \mathcal{K} .

1.2. The Hamiltonian algebra $\mathbf{H}(2n; \underline{1})$. Assume that $\text{char}(\mathcal{K}) = p$, then by definition (see [24]), the Jacobson-Witt algebra $\mathbf{W}(2n; \underline{1})$ is a restricted simple Lie algebra over a field \mathcal{K} . Its structure of p -Lie algebra is given by $D^{[p]} = D^p$, $\forall D \in \mathbf{W}(2n; \underline{1})$ with a basis $\{x^{(\alpha)} D_j \mid -n \leq j \leq n, j \neq 0, 0 \leq \alpha \leq \tau\}$, where $\tau = (p-1, \dots, p-1) \in \mathbb{N}^{2n}$; $\epsilon_i = (\delta_{-1,i}, \dots, \delta_{-n,i}, \delta_{1,i}, \dots, \delta_{n,i})$ such that $x^{(\epsilon_i)} = x_i$; and $\mathcal{O}(2n; \underline{1}) := \{x^{(\alpha)} \mid 0 \leq \alpha \leq \tau\}$ is the restricted divided power algebra with $x^{(\alpha)} x^{(\beta)} = \binom{\alpha + \beta}{\alpha} x^{\alpha + \beta}$ and a convention: $x^{(\alpha)} = 0$ if α has a component $\alpha_j < 0$ or $\geq p$, where $\binom{\alpha + \beta}{\alpha} = \prod_{i=1}^n \binom{\alpha_i + \beta_i}{\alpha_i} \binom{\alpha_{-i} + \beta_{-i}}{\alpha_{-i}}$. Note that

$\mathcal{O}(2n; \underline{1})_i = \text{Span}_{\mathcal{K}}\{x^{(\alpha)} \mid 0 \leq \alpha \leq \tau, |\alpha| = i\}$ where $|\alpha| = \sum_{i=1}^n (a_i + a_{-i})$. Moreover, $\mathbf{W}(2n; \underline{1})$ is isomorphic to $\text{Der}_{\mathcal{K}}(\mathcal{O}(2n; \underline{1}))$ and inherits a gradation from $\mathcal{O}(2n; \underline{1})$ by means of $\mathbf{W}(2n; \underline{1})_i = \sum_{j=1}^n (\mathcal{O}(2n; \underline{1})_{i+1} D_j + \mathcal{O}(2n; \underline{1})_{i-1} D_{-j})$. Define $D_H : \mathcal{O}(2n; \underline{1}) \rightarrow \mathbf{W}(2n; \underline{1})$ as $D_H(x^{(\alpha)}) = \sum_{i=1}^n (x^{(\alpha - \epsilon_{-i})} D_i - x^{(\alpha - \epsilon_i)} D_{-i})$. Then the subspace $\mathbf{H}'(2n; \underline{1}) := D_H(\mathcal{O}(2n; \underline{1}))$ is a subalgebra of $\mathbf{W}(2n; \underline{1})$. Its derived algebra $\mathbf{H}(2n; \underline{1})$ is called the Hamiltonian algebra, and $\mathbf{H}(2n; \underline{1}) = \bigoplus_{i=-1}^s \mathbf{H}(2n; \underline{1})_i \cap \mathbf{W}(2n; \underline{1})_i$ is graded with $s = |\tau| - 3$. Then by Proposition 4.4.4 and Theorem 4.4.5 in [25], $\mathbf{H}(2n; \underline{1}) = \text{Span}_{\mathcal{K}}\{D_H(x^{(\alpha)}) \mid x^{(\alpha)} \in \mathcal{O}(2n; \underline{1}), 0 \leq \alpha < \tau\}$ is a p -subalgebra of $\mathbf{W}(2n; \underline{1})$ with restricted gradation.

By definition (cf. [25]), the restricted universal enveloping algebra $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$ is isomorphic to $U(\mathbf{H}(2n; \underline{1}))/I$, where I is the Hopf ideal of $U(\mathbf{H}(2n; \underline{1}))$ generated by $(D_H(x^{(\epsilon_i + \epsilon_{-i})}))^p - D_H(x^{(\epsilon_i + \epsilon_{-i})})$, $(D_H(x^{(\alpha)}))^p$ with $\alpha \neq \epsilon_i + \epsilon_{-i}$ for $1 \leq i \leq n$. Since $\dim_{\mathcal{K}} \mathbf{H}(2n; \underline{1}) = p^{2n} - 2$, we have $\dim_{\mathcal{K}} \mathbf{u}(\mathbf{H}(2n; \underline{1})) = p^{p^{2n}-2}$.

1.3. Quantization by Drinfel'd twists. The following result is well-known (see [1]).

LEMMA 1.1. *Let $(A, m, \iota, \Delta_0, \varepsilon, S_0)$ be a Hopf algebra over a commutative ring. A Drinfel'd twist \mathcal{F} on A is an invertible element of $A \otimes A$ that:*

$$\begin{aligned} (\mathcal{F} \otimes 1)(\Delta_0 \otimes \text{Id})(\mathcal{F}) &= (1 \otimes \mathcal{F})(\text{Id} \otimes \Delta_0)(\mathcal{F}), \\ (\varepsilon \otimes \text{Id})(\mathcal{F}) &= 1 = (\text{Id} \otimes \varepsilon)(\mathcal{F}). \end{aligned}$$

Then, $w = m(\text{Id} \otimes S_0)(\mathcal{F})$ is invertible in A with $w^{-1} = m(S_0 \otimes \text{Id})(\mathcal{F}^{-1})$.

Moreover, if we define $\Delta : A \rightarrow A \otimes A$ and $S : A \rightarrow A$ by

$$\Delta(a) = \mathcal{F} \Delta_0 \mathcal{F}^{-1}, \quad S(a) = w S_0(a) w^{-1},$$

then $(A, m, \iota, \Delta, \varepsilon, S)$ is a new Hopf algebra, called the twisting of A by the Drinfel'd twist \mathcal{F} .

Let $\mathbb{F}[[t]]$ be a ring of formal power series over a field \mathbb{F} with $\text{char } \mathbb{F} = 0$. Assume that L is a triangular Lie algebra over \mathbb{F} with a classical Yang-Baxter r -matrix r (see [3, 8]). Let $U(L)$ denote the universal enveloping algebra of L , with the standard Hopf algebra $(U(L), m, \iota, \Delta_0, \varepsilon, S_0)$.

Let us consider the topologically free $\mathbb{F}[[t]]$ -algebra $U(L)[[t]]$ (for the definition, see p.4 of [8]), which can be viewed as an associative \mathbb{F} -algebra of formal power series with coefficients in $U(L)$. Naturally, $U(L)[[t]]$ equips with an induced Hopf algebra structure arising from that on $U(L)$. By abuse of notation, we denote it by $(U(L)[[t]], m, \iota, \Delta_0, \varepsilon, S_0)$.

DEFINITION 1.2. [13] For a triangular Lie algebra L over \mathbb{F} with $\text{char}(\mathbb{F}) = 0$, $U(L)[[t]]$ is called a *quantization* of $U(L)$ by a Drinfel'd twist \mathcal{F} over $U(L)[[t]]$ if

$U(L)[[t]]/tU(L)[[t]] \cong U(L)$, and \mathcal{F} is determined by its r -matrix r (namely, its Lie bialgebra structure).

2. Drinfel'd twists in $U(\mathbf{H})[[t]]$

2.1. Construction of Drinfel'd twists. Let L be a Lie algebra containing linearly independent elements h and e satisfying $[h, e] = e$; then the classical Yang-Baxter r -matrix $r = h \otimes e - e \otimes h$ equips L with the structure of a triangular coboundary Lie bialgebra (see [17]). To describe a quantization of $U(L)$ by a Drinfel'd twist \mathcal{F} over $U(L)[[t]]$, we need an explicit construction for such a Drinfel'd twist. In what follows, we shall see that such a Drinfel'd twist depends on the choice of two distinguished elements h and e arising from its r -matrix r .

For any element of a unital R -algebra (R a ring) and $a \in R$, we set

$$\begin{aligned} x_a^{\langle m \rangle} &:= (x + a)(x + a + 1) \cdots (x + a + m - 1); \\ x_a^{[m]} &:= (x + a)(x + a - 1) \cdots (x + a - m + 1), \end{aligned}$$

and then denote $x^{\langle m \rangle} := x_0^{\langle m \rangle}$, $x^{[m]} := x_0^{[m]}$.

Note that h and e satisfy the following equalities:

$$\begin{aligned} e^s \cdot h_a^{[m]} &= h_{a-s}^{[m]} \cdot e^s, \\ e^s \cdot h_a^{\langle m \rangle} &= h_{a-s}^{\langle m \rangle} \cdot e^s, \end{aligned}$$

where m, s are non-negative integers, $a \in \mathbb{F}$.

For $a \in \mathbb{F}$, we set $\mathcal{F}_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_a^{[r]} \otimes e^r t^r$, $F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{\langle r \rangle} \otimes e^r t^r$, $u_a = m \cdot (S_0 \otimes \text{Id})(F_a)$, $v_a = m \cdot (\text{Id} \otimes S_0)(\mathcal{F}_a)$. Write $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, $u = u_0$, $v = v_0$. Since $S_0(h_a^{\langle r \rangle}) = (-1)^r h_{-a}^{[r]}$ and $S_0(e^r) = (-1)^r e^r$, one has $v_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{[r]} e^r t^r$, $u_b = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-b}^{[r]} e^r t^r$.

LEMMA 2.1. ([12]) For $a, b \in \mathbb{F}$, one has

$$\mathcal{F}_a F_b = 1 \otimes (1 - et)^{a-b}, \quad \text{and} \quad v_a u_b = (1 - et)^{-(a+b)}.$$

COROLLARY 2.2. For $a \in \mathbb{F}$, \mathcal{F}_a and u_a are invertible with $\mathcal{F}_a^{-1} = F_a$ and $u_a^{-1} = v_{-a}$. In particular, $\mathcal{F}^{-1} = F$ and $u^{-1} = v$.

LEMMA 2.3. ([13]) For any positive integers r , we have

$$\Delta_0(h^{[r]}) = \sum_{i=0}^r \binom{r}{i} h^{[i]} \otimes h^{[r-i]}.$$

Furthermore, $\Delta_0(h^{[r]}) = \sum_{i=0}^r \binom{r}{i} h_{-s}^{[i]} \otimes h_s^{[r-i]}$ for any $s \in \mathbb{F}$.

PROPOSITION 2.4. ([12, 13]) *If a Lie algebra L contains a two-dimensional solvable Lie subalgebra with a basis $\{h, e\}$ satisfying $[h, e] = e$, then $\mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]} \otimes e^r t^r$ is a Drinfel'd twist on $U(L)[[t]]$.*

2.2. Basic Drinfel'd twists. Take two distinguished elements $e = D_H(x^\alpha)$, $h = D_H(x^{\epsilon_i + \epsilon_{-i}})$ such that $[h, e] = e$, where $1 \leq i \leq n$. It is easy to see that $\alpha_i - \alpha_{-i} = 1$. Using the result of [17], we have the following

PROPOSITION 2.5. *There is a triangular Lie bialgebra structure on \mathbf{H} given by the classical Yang-Baxter r -matrix*

$$r := D_H(x^{\epsilon_i + \epsilon_{-i}}) \otimes D_H(x^\alpha) - D_H(x^\alpha) \otimes D_H(x^{\epsilon_i + \epsilon_{-i}}), \quad 1 \leq i \leq n,$$

where $\alpha \in \mathbb{Z}^{2n}$, $\alpha_i - \alpha_{-i} = 1$, and $[D_H(x^{\epsilon_i - \epsilon_{-i}}), D_H(x^\alpha)] = D_H(x^\alpha)$.

Fix two distinguished elements $h = D_H(x^{\epsilon_i + \epsilon_{-i}})$, $e = D_H(x^\alpha)$, with $\alpha_i - \alpha_{-i} = 1$, then $\mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]} \otimes e^r t^r$ is a Drinfel'd twist on $U(\mathbf{H})[[t]]$. But the coefficients of the quantizations of the standard Hopf structure $(U(\mathbf{H})[[t]], m, \iota, \Delta_0, S_0, \varepsilon)$ by \mathcal{F} may be not integral. In order to get integral forms, it suffices to consider what conditions are needed for those coefficients to be integers.

LEMMA 2.6. ([12]) *For any $a, k, \ell \in \mathbb{Z}$, $a^\ell \prod_{j=0}^{\ell-1} (k+ja)/\ell!$ is an integer.* \square

From this lemma, we are interested in the following two simple cases:

- (i) $h = D_H(x^{\epsilon_k + \epsilon_{-k}})$, $e = D_H(x^{2\epsilon_k + \epsilon_{-k}})$ ($1 \leq k \leq n$);
- (ii) $h = D_H(x^{\epsilon_k + \epsilon_{-k}})$, $e = D_H(x^{\epsilon_k + \epsilon_m})$ ($m \neq k, -k$).

Let $\mathcal{F}(k)$ be the corresponding Drinfel'd twist in case (i) and $\mathcal{F}(k; m)$ the corresponding Drinfel'd twist in case (ii).

DEFINITION 2.7. $\mathcal{F}(k)$ ($1 \leq k \leq n$) are called *vertical basic Drinfel'd twists*, $\mathcal{F}(k; m)$ ($1 \leq k, m \leq n, m \neq k, -k$) are called *horizontal basic Drinfel'd twists*.

REMARK 2.8. In case (i): we get n vertical basic Drinfel'd twists $\mathcal{F}(1), \dots, \mathcal{F}(n)$ over $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$. It is interesting to consider the products of some basic Drinfel'd twists, one can get many more new Drinfel'd twists which will lead to many more new complicated quantizations not only over the $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$, but also over the $\mathbf{u}(\mathbf{H}(n; \underline{1}))$ as well, via our modulo reduction approach developed in the next section.

In case (ii): according to the parametrization of twists $\mathcal{F}(k; m)$, we get $2n(n-1)$ horizontal basic Drinfel'd twists over $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$. We will discuss these twists and corresponding quantizations in Section 4.

2.3. More Drinfel'd twists. We consider the products of pairwise different and mutually commutative basic Drinfel'd twists and can get many more new complicated quantizations not only over the $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$, but over the $\mathbf{u}(\mathbf{H}(n; \underline{1}))$ as well. We can treat the same question as Cartan type S Lie algebra in [14], and get the following results.

THEOREM 2.9. $\mathcal{F}(k)\mathcal{F}(k')$ ($1 \leq k \neq k' \leq n$) is still a Drinfel'd twist on $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$.

More generally, we have the following

COROLLARY 2.10. Let $\mathcal{F}(k_1), \dots, \mathcal{F}(k_m)$ be m pairwise different basic Drinfel'd twists and $[\mathcal{F}(k_i), \mathcal{F}(k_s)] = 0$ for all $1 \leq i \neq s \leq m$. Then $\mathcal{F}(k_1) \cdots \mathcal{F}(k_m)$ is still a Drinfel'd twist.

We denote $\mathcal{F}_m = \mathcal{F}(k_1) \cdots \mathcal{F}(k_m)$ and its length as m . We shall show that the twisted structures given by Drinfel'd twists with different product-length are nonisomorphic.

DEFINITION 2.11. ([14]) A Drinfel'd twist $\mathcal{F} \in A \otimes A$ on any Hopf algebra A is called *compatible* if \mathcal{F} commutes with the coproduct Δ_0 .

In other words, twisting a Hopf algebra A with a *compatible* twist \mathcal{F} gives exactly the same Hopf structure, that is, $\Delta_{\mathcal{F}} = \Delta_0$. The set of *compatible* twists on A thus forms a group.

LEMMA 2.12. ([11]) Let $\mathcal{F} \in A \otimes A$ be a Drinfel'd twist on a Hopf algebra A . Then the twisted structure induced by \mathcal{F} coincides with the structure on A if and only if \mathcal{F} is a compatible twist.

LEMMA 2.13. ([14]) Let $\mathcal{F}, \mathcal{G} \in A \otimes A$ be Drinfel'd twists on a Hopf algebra A with $\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F}$ and $\mathcal{F} \neq \mathcal{G}$. Then $\mathcal{F}\mathcal{G}$ is a Drinfel'd twist. Furthermore, \mathcal{G} is a Drinfel'd twist on $A_{\mathcal{F}}$, \mathcal{F} is a Drinfel'd twist on $A_{\mathcal{G}}$ and $\Delta_{\mathcal{F}\mathcal{G}} = (\Delta_{\mathcal{F}})_{\mathcal{G}} = (\Delta_{\mathcal{G}})_{\mathcal{F}}$.

PROPOSITION 2.14. Drinfel'd twists $\mathcal{F}^{\zeta(i)} := \mathcal{F}(1)^{\zeta_1} \cdots \mathcal{F}(n)^{\zeta_n}$ (where $\zeta(i) = (\zeta_1, \dots, \zeta_n) = (\underbrace{1, \dots, 1}_i, 0, \dots, 0) \in \mathbb{Z}_2^n$) lead to n different twisted Hopf algebra structures on $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$.

3. Quantizations of vertical type for the Lie bialgebra of Cartan type \mathbf{H}

In this section, we explicitly quantize the Lie bialgebras of \mathbf{H} by the vertical basic Drinfel'd twists, and obtain certain quantizations of the restricted universal enveloping algebra $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$ by the modular reduction and base changes.

3.1. Quantizations integral forms of the \mathbb{Z} -form $\mathbf{H}_{\mathbb{Z}}^+$ in characteristic 0. For the universal enveloping algebra $U(\mathbf{H})$ for the Lie algebra \mathbf{H} over \mathbb{F} , denote by $(U(\mathbf{H}), m, \iota, \Delta_0, S_0, \varepsilon)$ the standard Hopf algebra structure. We can perform the process of twisting the standard Hopf structure by the vertical Drinfel'd twist $\mathcal{F}(k)$ with $h := D_H(x^{\epsilon_k + \epsilon_{-k}}), e := D_H(x^{2\epsilon_k + \epsilon_{-k}})$. To simplify the formulas, let us introduce the operator $d^{(\ell)}$ on $U(\mathbf{H})$ defined by $d^{(\ell)} = \frac{1}{\ell!}(\text{ad } e)^\ell$. Use induction on ℓ , we can get $d^{(\ell)}(D_H(x^\alpha)) = A_\ell D_H(x^{\alpha + \ell\epsilon_k})$, where $A_\ell = \frac{1}{\ell!} \prod_{j=0}^{\ell-1} (\alpha_k - 2\alpha_{-k} + j)$ and set $A_0 = 1, A_{-1} = 0$.

Recall the vertical basic twist of Cartan type \mathbf{S} Lie algebra in [14] is given by $h = \partial_k - \partial_{-k}, e = x^{\epsilon_k}(\partial_k - 2\partial_{-k})$ (or the special algebra is given by $h = D_{k,-k}(x^{\epsilon_k + \epsilon_{-k}}), e = 2D_{k,-k}(x^{2\epsilon_k + \epsilon_{-k}})$), using the quantizations of Cartan type \mathbf{S} Lie algebra and $D_H(x^\alpha) = \sum_{i=1}^n D_{i,-i}(x^{(\alpha)}) = \sum_{i=1}^n x^{\alpha - \epsilon_i - \epsilon_{-i}}(\alpha_{-i}\partial_i - \alpha_i\partial_{-i})$, we have the following theorem which gives the quantization of $U(\mathbf{H})$ by Drinfel'd twist $\mathcal{F}(k)$.

THEOREM 3.1. *For the given two distinguished elements $h = D_H(x^{\epsilon_k + \epsilon_{-k}}), e = D_H(x^{2\epsilon_k + \epsilon_{-k}})$, such that $[h, e] = e$ in the generalized Cartan type H Lie algebra \mathbf{H} over \mathbb{F} , there exists a structure of noncommutative and noncocommutative Hopf algebra $(U(\mathbf{H})[[t]], m, \iota, \Delta, S, \varepsilon)$ on $U(\mathbf{H})[[t]]$ over $\mathbb{F}[[t]]$ with $U(\mathbf{H})[[t]]/tU(\mathbf{H})[[t]] \cong U(\mathbf{H})$, which leaves the product of $U(\mathbf{H})[[t]]$ undeformed but with the deformed coproduct, antipode and counit defined by*

$$\begin{aligned} \Delta(D_H(x^\alpha)) &= D_H(x^\alpha) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{\infty} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} \cdot d^{(\ell)}(D_H(x^\alpha)) t^\ell, \\ S(D_H(x^\alpha)) &= -(1-et)^{-(\alpha_k - \alpha_{-k})} \cdot \left(\sum_{\ell=0}^{\infty} d^{(\ell)}(D_H(x^\alpha)) \cdot h_1^{(\ell)} t^\ell \right), \\ \varepsilon(D_H(x^\alpha)) &= 0, \end{aligned}$$

where $D_H(x^\alpha) \in \mathbf{H}$.

As is known, $\{D_H(x^\alpha) | \alpha \in \mathbb{Z}_+^{2n} \setminus \{0\}\}$ is a \mathbb{Z} -basis of $\mathbf{H}_{\mathbb{Z}}^+$, as a subalgebra of $\mathbf{H}_{\mathbb{Z}}$ and $\mathbf{W}_{\mathbb{Z}}^+$. As a result of the Theorem, we have:

COROLLARY 3.2. For the given two distinguished elements $h := D_H(x^{\epsilon_k + \epsilon_{-k}}), e := D_H(x^{2\epsilon_k + \epsilon_{-k}})$ ($1 \leq k \leq n$), the corresponding quantization of $U(\mathbf{H}_{\mathbb{Z}}^+)$ over $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$ by the Drinfel'd twist $\mathcal{F}(k)$ with the product undeformed is given by:

$$\begin{aligned} \Delta(D_H(x^\alpha)) &= D_H(x^\alpha) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{\infty} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} \cdot A_\ell(D_H(x^{\alpha + \ell\epsilon_k})) t^\ell, \\ S(D_H(x^\alpha)) &= -(1-et)^{-(\alpha_k - \alpha_{-k})} \cdot \left(\sum_{\ell=0}^{\infty} A_\ell(D_H(x^{\alpha + \ell\epsilon_k})) \cdot h_1^{(\ell)} t^\ell \right), \\ \varepsilon(D_H(x^\alpha)) &= 0, \end{aligned}$$

where $A_\ell = \frac{1}{\ell!} \prod_{j=0}^{\ell-1} (\alpha_k - 2\alpha_{-k} + j)$, with $A_0 = 1, A_{-1} = 0$.

3.2. Quantizations of the Hamiltonian algebra $\mathbf{H}(2n, \underline{1})$. In this subsection, firstly, we use the quantization of $U(\mathbf{H}_{\mathbb{Z}}^+)$ in characteristic 0 (Corollary 3.2) to yield the quantization of $U(\mathbf{H}(2n; \underline{1}))$, for the restricted simple modular Lie algebra $\mathbf{H}(2n; \underline{1})$ in characteristic p . Secondly, we use the quantization of $U(\mathbf{H}(2n; \underline{1}))$ to yield the required quantization of $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$, the restricted universal enveloping algebra of $\mathbf{H}(2n; \underline{1})$.

Let \mathbb{Z}_p be the prime subfield of \mathcal{K} with $\text{char}(\mathcal{K}) = p$. When considering $\mathbf{W}_{\mathbb{Z}}^+$ as a \mathbb{Z}_p -Lie algebra, namely, making modulo p reduction for the defining relations of $\mathbf{W}_{\mathbb{Z}}^+$, denoted by $\mathbf{W}_{\mathbb{Z}_p}^+$, we see that $(J_{\underline{1}})_{\mathbb{Z}_p} = \text{Span}_{\mathbb{Z}_p} \{x^\alpha D_i \mid \exists j : \alpha_j \geq p\}$ is a maximal ideal of $\mathbf{W}_{\mathbb{Z}_p}^+$, and $\mathbf{W}_{\mathbb{Z}_p}^+ / (J_{\underline{1}})_{\mathbb{Z}_p} \cong \mathbf{W}(n; \underline{1})_{\mathbb{Z}_p} = \text{Span}_{\mathbb{Z}_p} \{x^{(\alpha)} D_i \mid 0 \leq \alpha \leq \tau, 1 \leq i \leq n\}$. For the subalgebra $\mathbf{H}_{\mathbb{Z}}^+$, we have $\mathbf{H}_{\mathbb{Z}_p}^+ / (\mathbf{H}_{\mathbb{Z}_p}^+ \cap (J_{\underline{1}})_{\mathbb{Z}_p}) \cong \mathbf{H}'(2n; \underline{1})_{\mathbb{Z}_p}$. We denote $\mathbf{H}_{\mathbb{Z}_p}^+ \cap (J_{\underline{1}})_{\mathbb{Z}_p}$ simply as $(J_{\underline{1}}^+)_{\mathbb{Z}_p}$.

Moreover, we have $\mathbf{H}'(2n; \underline{1}) = \mathcal{K} \otimes_{\mathbb{Z}_p} \mathbf{H}'(2n; \underline{1})_{\mathbb{Z}_p} = \mathcal{K} \mathbf{H}'(2n; \underline{1})_{\mathbb{Z}_p}$, and $\mathbf{H}_{\mathcal{K}}^+ = \mathcal{K} \mathbf{H}_{\mathbb{Z}_p}^+$.

Observe that the ideal $J_{\underline{1}}^+ := \mathcal{K}(J_{\underline{1}}^+)_{\mathbb{Z}_p}$ generates an ideal of $U(\mathbf{H}_{\mathcal{K}}^+)$ over \mathcal{K} , denoted by $J := J_{\underline{1}}^+ U(\mathbf{H}_{\mathcal{K}}^+)$, where $\mathbf{H}_{\mathcal{K}}^+ / J_{\underline{1}}^+ \cong \mathbf{H}'(2n; \underline{1})$. Based on the formulae of Corollary 3.2, J is a Hopf ideal of $U(\mathbf{H}_{\mathcal{K}}^+)$ satisfying $U(\mathbf{H}_{\mathcal{K}}^+) / J \cong U(\mathbf{H}'(2n; \underline{1}))$. Note that elements $\sum a_{i,\alpha} \frac{1}{\alpha!} D_H(x^\alpha)$ in $\mathbf{H}_{\mathcal{K}}^+$ for $0 \leq \alpha \leq \tau$ will be identified with $\sum a_{i,\alpha} D_H(x^{(\alpha)})$ in $\mathbf{H}'(2n; \underline{1})$ and those in $J_{\underline{1}}$ with 0. Hence, by Corollary 3.2, we get the quantization of $U(\mathbf{H}'(2n; \underline{1}))$ over $U_t(\mathbf{H}'(2n; \underline{1})) := U(\mathbf{H}'(2n; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]$ (not necessarily in $U(\mathbf{H}'(2n; \underline{1}))[[t]]$, as seen in formulae 3.1 & 3.2 as follows.

THEOREM 3.3. *For the given two distinguished elements $h = D_H(x^{(\epsilon_k + \epsilon_{-k})})$, $e := 2D_H(x^{(2\epsilon_k + \epsilon_{-k})})$ ($1 \leq k \leq n$), the corresponding quantization of $U(\mathbf{H}'(2n; \underline{1}))$ over $U_t(\mathbf{H}'(2n; \underline{1}))$ with the product undeformed is given by*

(3.1)

$$\Delta(D_H(x^{(\alpha)})) = D_H(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} \bar{A}_\ell D_H(x^{(\alpha + \ell \epsilon_k)}) h_1^{(\ell)} t^\ell,$$

$$(3.2) \quad S(D_H(x^{(\alpha)})) = -(1-et)^{\alpha_{-k} - \alpha_k} \sum_{\ell=0}^{p-1} \bar{A}_\ell D_H(x^{(\alpha + \ell \epsilon_k)}) h_1^{(\ell)} t^\ell,$$

$$(3.3) \quad \varepsilon(D_H(x^{(\alpha)})) = 0,$$

where $\bar{A}_\ell = \ell! \binom{\alpha_k + \ell}{\alpha_k} A_\ell \pmod{p}$, with $A_\ell = \frac{1}{\ell!} \prod_{j=0}^{\ell-1} (\alpha_k - 2\alpha_{-k} + j)$, $A_0 = 1$ and $A_{-1} = 0$.

Note that the formula of the Theorem 3.3, when $\alpha + \ell \epsilon_k = \tau, \epsilon_{-k} = p-1$, $\alpha_k - 2\alpha_{-k} + \ell - 1 = 0 \pmod{p}$, i.e., $\bar{A}_\ell = 0$, so it also give the corresponding quantization of $U(\mathbf{H}(2n; \underline{1}))$ over $U_t(\mathbf{H}(2n; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]$ and also over $U(\mathbf{H}(2n; \underline{1}))[[t]]$. It should be noticed that in this step — including from the quantization integral form

of $U(\mathbf{H}_{\mathbb{Z}}^+)$ and making the modulo p reduction, we used the first base change with $\mathcal{K}[[t]]$ by $\mathcal{K}[t]$, and the objects from $U(\mathbf{H}(2n; \underline{1}))[[t]]$ turning to $U_t(\mathbf{H}(2n; \underline{1}))$.

Denote by I the ideal of $U(\mathbf{H}(2n; \underline{1}))$ over \mathcal{K} generated by $(D_H(x^{(\alpha)}))^p$ with $\alpha \neq \epsilon_i + \epsilon_{-i}$ for $0 < \alpha < \tau$ and $(D_H(x^{(\epsilon_i + \epsilon_{-i})}))^p - D_H(x^{(\epsilon_i + \epsilon_{-i})})$ for $1 \leq i \leq n$. $\mathbf{u}(\mathbf{H}(2n; \underline{1})) = U(\mathbf{H}(2n; \underline{1}))/I$ is of prime-power dimension p^{2n-2} . In order to get a reasonable quantization of prime-power dimension for $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$ in characteristic p , it should start to be induced from the $\mathcal{K}[t]$ -algebra $U_t(\mathbf{H}(2n; \underline{1}))$ in Theorem 3.3.

By the proof of [14], we have the following results:

- LEMMA 3.4. (i) $(1 - et)^p \equiv 1 \pmod{p, I}$.
(ii) $(1 - et)^{-1} \equiv 1 + et + \dots + e^{p-1}t^{p-1} \pmod{p, I}$.
(iii) $h_a^{(\ell)} \equiv 0 \pmod{p, I}$ for $\ell \geq p$, and $a \in \mathbb{Z}_p$.

The above Lemma, together with Theorem 3.3, indicates that the required t -deformation of $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$ (if it exists) in fact only happens in a p -truncated polynomial ring (with degrees of t less than p) with coefficients in $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$, i.e., $\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})) := \mathbf{u}(\mathbf{H}(2n; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]_p^{(q)}$ (rather than in $\mathbf{u}_t(\mathbf{H}(2n; \underline{1})) := \mathbf{u}(\mathbf{H}(2n; \underline{1})) \otimes_{\mathcal{K}} \mathcal{K}[t]$), where $\mathcal{K}[t]_p^{(q)}$ is conveniently taken to be a p -truncated polynomial ring which is a quotient of $\mathcal{K}[t]$ defined as

$$\mathcal{K}[t]_p^{(q)} = \mathcal{K}[t]/(t^p - qt), \quad \text{for } q \in \mathcal{K}.$$

Thereby, we obtain the underlying ring for our required t -deformation of $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$ over $\mathcal{K}[t]_p^{(q)}$, and $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})) = p \cdot \dim_{\mathcal{K}} \mathbf{u}(\mathbf{H}(2n; \underline{1})) = p^{2n-1}$. Via modulo “restrictedness” reduction, it is necessary for us to work over the objects from $U_t(\mathbf{H}(2n; \underline{1}))$ passage to $U_{t,q}(\mathbf{H}(2n; \underline{1}))$ first, and then to $\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1}))$ (see the proof of Theorem 3.7 below), here we used the second base change with $\mathcal{K}[t]_p^{(q)}$ instead of $\mathcal{K}[t]$.

As the definition of [14], we gave the following description:

DEFINITION 3.5. With notation as above. A Hopf algebra $(\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})), m, \iota, \Delta, S, \varepsilon)$ over a ring $\mathcal{K}[t]_p^{(q)}$ of characteristic p is said to be a finite-dimensional quantization of $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$ if its Hopf algebra structure, via modular reduction and base changes, inherits from a twisting of the standard Hopf algebra $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$ by a Drinfel’d twist such that $\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1}))/t\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})) \cong \mathbf{u}(\mathbf{H}(2n; \underline{1}))$.

To describe $\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1}))$ explicitly, we still need an auxiliary lemma.

LEMMA 3.6. Let $e = 2D_H(x^{(2\epsilon_k + \epsilon_{-k})})$, and $d^{(\ell)} = \frac{1}{\ell!} \text{ad } e$, then:

- (i) $d^{(\ell)} D_H(x^{(\alpha)}) = \overline{A}_{\ell} D_H(x^{(\alpha + \ell \epsilon_k)})$, where \overline{A}_{ℓ} as in theorem 3.3.
(ii) $d^{(\ell)} D_H(x^{(\epsilon_i + \epsilon_{-i})}) = \delta_{\ell,0} D_H(x^{(\epsilon_i + \epsilon_{-i})}) - \delta_{\ell,1} \delta_{k,i} e$ for $1 \leq i \leq n$.
(iii) $d^{(\ell)} (D_H(x^{(\alpha)}))^p = \delta_{\ell,0} (D_H(x^{(\alpha)}))^p - \delta_{\ell,1} \delta_{\alpha, \epsilon_k + \epsilon_{-k}} e$.

Based on the Theorem 3.3, Definition 3.5 and Lemma 3.6, we arrive at:

THEOREM 3.7. *Fix the given two distinguished elements $h := D_H(x^{(\epsilon_k + \epsilon_{-k})})$, $e := 2D_H(x^{(2\epsilon_k + \epsilon_{-k})})$ ($1 \leq k \leq n$), there is a noncommutative and noncocommutative Hopf algebra $(\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})), m, \iota, \Delta, S, \varepsilon)$ over $\mathcal{K}[t]_p^{(q)}$ with its algebra structure undeformed, whose coalgebra structure is given by*

$$\begin{aligned} \Delta(D_H(x^{(\alpha)})) &= D_H(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)} D_H(x^{(\alpha)}) t^\ell, \\ S(D_H(x^{(\alpha)})) &= -(1-et)^{\alpha_k - \alpha_{-k}} \sum_{\ell=0}^{p-1} d^{(\ell)} (D_H(x^{(\alpha)})) h_1^{(\ell)} t^\ell, \\ \varepsilon(D_H(x^{(\alpha)})) &= 0, \end{aligned}$$

for $0 < \alpha < \tau$, which is a finite dimensional with $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})) = p^{p^{2n}-1}$.

REMARK 3.8. (i) Set $f = (1-et)^{-1}$. By Lemma 3.6 & Theorem 3.7, one gets

$$[h, f] = f^2 - f, \quad h^p = h, \quad f^p = 1, \quad \Delta(h) = h \otimes f + 1 \otimes h,$$

where f is a group-like element, and $S(h) = -hf^{-1}$, $\varepsilon(h) = 0$. So the subalgebra generated by h and f is a Hopf subalgebra of $\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1}))$, which is isomorphic to the well-known Radford Hopf algebra over \mathcal{K} in characteristic p (see [23]).

(ii) According to our argument, given a parameter $q \in \mathcal{K}$, one can specialize t to any root of the p -polynomial $t^p - qt \in \mathcal{K}[t]$ in a split field of \mathcal{K} . For instance, if take $q = 1$, then one can specialize t to any scalar in \mathbb{Z}_p . If set $t = 0$, then we get the original standard Hopf algebra structure of $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$. In this way, we indeed get a new Hopf algebra structure over the same restricted universal enveloping algebra $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$ over \mathcal{K} under the assumption that \mathcal{K} is algebraically closed, which has the new coalgebra structure induced by Theorem 3.7, but has dimension $p^{p^{2n}-1}$.

(iii) We can consider the modular reduction process for the quantizations of $U(\mathbf{H}^+)[[t]]$ arising from those products of some pairwise different and mutually commutative basic Drinfel'd twists. We will then get lots of new families of noncommutative and noncocommutative Hopf algebras of dimension $p^{p^{2n}-1}$ with indeterminate t or of dimension $p^{p^{2n}-2}$ with specializing t into a scalar in \mathcal{K} .

4. Quantizations of horizontal type for Lie bialgebra of Cartan type \mathbf{H}

4.1. Quantizations of horizontal type of $\mathbf{u}(\mathbf{H}(2n; \underline{1}))$. In this section, we assume that $n \geq 2$. Take $h := D_H(x^{\epsilon_k + \epsilon_{-k}})$ and $e := D_H(x^{\epsilon_k + \epsilon_m})$, ($1 \leq k, |m| \leq n, m \neq \pm k$) and denote by $\mathcal{F}(k; m)$ the corresponding Drinfel'd twist. Set $d^{(\ell)} = \frac{1}{\ell!}(\text{ad } e)^\ell$. For $m \in \{-1, \dots, -n, 1, \dots, n\}$, set $\sigma(m) := \begin{cases} 1, & m < 0, \\ -1, & m > 0. \end{cases}$ Using the horizontal Drinfel'd twists, we will obtain some new quantizations of horizontal type for the universal enveloping algebra of the Hamiltonian algebra $\mathbf{H}(2n; \underline{1})$. The twisted structures given by the twists $\mathcal{F}(k; m)$ on subalgebra $\mathbf{H}(2n; \underline{1})_0$ are the same

as those on the symplectic Lie algebra \mathfrak{sp}_{2n} over a field \mathcal{K} with $\text{char}(\mathcal{K}) = p$ derived by the Jordanian twists $\mathcal{F} = \exp(h \otimes \sigma)$, $\sigma = \ln(1-e)$ for some two-dimensional carrier subalgebra $B(2) = \text{Span}_{\mathcal{K}}\{h, e\}$ discussed in [15, 16], etc.

LEMMA 4.1. *For $h := D_H(x^{\epsilon_k + \epsilon_{-k}})$ and $e := D_H(x^{\epsilon_k + \epsilon_m})$, ($1 \leq k, |m| \leq n$, $m \neq \pm k$), and $a \in \mathbb{F}$, $D_H(x^\alpha)$, $a_i \in \mathbf{H}$, the following equalities hold in $U(\mathbf{H})$:*

$$\begin{aligned}
\text{(i)} \quad & D_H(x^\alpha) \cdot h_a^{(s)} = h_{a+(\alpha_{-k}-\alpha_k)}^{(s)} \cdot D_H(x^\alpha), \\
\text{(ii)} \quad & D_H(x^\alpha) \cdot h_a^{[s]} = h_{a+(\alpha_{-k}-\alpha_k)}^{[s]} \cdot D_H(x^\alpha), \\
\text{(iii)} \quad & d^{(\ell)}(D_H(x^\alpha)) = \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}), \\
\text{(iv)} \quad & d^{(\ell)}(a_1 \cdots a_s) = \sum_{\ell_1+\cdots+\ell_s=\ell} d^{\ell_1}(a_1) \cdots d^{\ell_s}(a_s), \\
\text{(v)} \quad & D_H(x^\alpha) \cdot e^s = \sum_{\ell=0}^s (-1)^\ell \binom{s}{\ell} e^{s-\ell} (\text{ad } e)^\ell (D_H(x^\alpha)) \\
& = \sum_{\ell=0}^s (-1)^\ell \ell! \binom{s}{\ell} e^{s-\ell} \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}),
\end{aligned}$$

where $A_j = \frac{1}{j!} \prod_{i=0}^{j-1} (\alpha_{-k} - i)$, $B_j = \sigma(m) j \frac{1}{j!} \prod_{i=0}^{j-1} (\alpha_{-m} - i)$, with $A_0 = B_0 = 1$, $A_j = 0$, for $j > \alpha_{-k}$, $B_j = 0$ for $j > \alpha_{-m}$.

PROOF. We only need to prove (iii), the proof of the other relations is the same as in [14].

For (iii), use induction on ℓ . This holds for $\ell = 1$ since

$$\begin{aligned}
d(D_H(x^\alpha)) &= [D_H(x^{\epsilon_k + \epsilon_m}), D_H(x^\alpha)] \\
&= \sum_{i=1}^n \partial_{-i}(\epsilon_k + \epsilon_m) \alpha_i - \partial_i(\epsilon_k + \epsilon_m) \alpha_{-i} D_H(x^{\alpha+\epsilon_k+\epsilon_m-\epsilon_i-\epsilon_{-i}}) \\
&= \begin{cases} -\alpha_{-m} D_H(x^{\alpha+\epsilon_k-\epsilon_{-m}}) - \alpha_{-k} D_H(x^{\alpha+\epsilon_m-\epsilon_{-k}}), & m > 0 \\ \alpha_{-m} D_H(x^{\alpha+\epsilon_k-\epsilon_{-m}}) - \alpha_{-k} D_H(x^{\alpha+\epsilon_m-\epsilon_{-k}}), & m < 0 \end{cases} \\
&= \sigma(m) \alpha_{-m} D_H(x^{\alpha+\epsilon_k-\epsilon_{-m}}) - \alpha_{-k} D_H(x^{\alpha+\epsilon_m-\epsilon_{-k}}).
\end{aligned}$$

For $\ell \geq 1$, we have

$$\begin{aligned}
d^{(\ell+1)} D_H(x^\alpha) &= \frac{\text{ad } e}{\ell+1} \cdot \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) \\
&= \frac{1}{\ell+1} \cdot \sum_{j=0}^{\ell} A_j B_{\ell-j} \left(\sigma(m) (\alpha_{-m} - (\ell-j)) D_H(x^{\alpha+(\ell-j+1)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) \right. \\
&\quad \left. - (\alpha_{-k} - j) D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+(j+1)(\epsilon_m-\epsilon_{-k})}) \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{\ell} \frac{\ell-j+1}{\ell+1} A_j B_{\ell-j+1} D_H(x^{\alpha+(\ell-j+1)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) \\
&\quad + \sum_{j=1}^{\ell+1} \frac{j}{\ell+1} A_j B_{\ell-j+1} D_H(x^{\alpha+(\ell-j+1)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) \\
&= B_{\ell+1} D_H(x^{\alpha+(\ell+1)(\epsilon_k-\epsilon_{-m})}) \\
&\quad + \sum_{j=1}^{\ell} A_j B_{\ell-j+1} \left(\frac{\ell-j+1}{\ell+1} + \frac{j}{\ell+1} \right) D_H(x^{\alpha+(\ell-j+1)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) \\
&\quad + A_{\ell+1} D_H(x^{\alpha+(\ell+1)(\epsilon_m-\epsilon_{-k})}) \\
&= \sum_{j=0}^{\ell+1} A_j B_{\ell+1-j} D_H(x^{\alpha+(\ell+1-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}).
\end{aligned}$$

This completes the proof. \square

LEMMA 4.2. For $h := D_H(x^{\epsilon_k+\epsilon_{-k}})$ and $e := D_H(x^{\epsilon_k+\epsilon_m})$, ($1 \leq k, |m| \leq n$, $m \neq \pm k$), and $a \in \mathbb{F}$, $D_H(x^\alpha) \in \mathbf{H}$, the following equalities hold in $U(\mathbf{H})$:

$$\begin{aligned}
\text{(i)} \quad & (\text{ad } D_H(x^\alpha))^s \cdot e = \sum_{i=0}^s (-\sigma(m))^i \binom{s}{i} A(s-i-1, k) A(i-1, m) \times \\
& \quad \times D_H(x^{s\alpha-i(\epsilon_m+\epsilon_{-m})-(s-i)(\epsilon_k+\epsilon_{-k})+\epsilon_k+\epsilon_m}); \\
\text{(ii)} \quad & ((D_H(x^\alpha))^s \otimes 1) \cdot F_a = F_{a+s(\alpha_{-k}-\alpha_k)} \cdot ((D_H(x^\alpha))^s \otimes 1); \\
\text{(iii)} \quad & (D_H(x^\alpha))^s \cdot u_a = u_{a+s(\alpha_k-\alpha_{-k})} \left(\sum_{\ell=0}^{\infty} d^{(\ell)}((D_H(x^\alpha))^s) \cdot h_{1-a}^{(\ell)} t^\ell \right); \\
\text{(iv)} \quad & (1 \otimes (D_H(x^\alpha))^s) \cdot F_a = \sum_{\ell=0}^{\infty} (-1)^\ell F_{a+\ell} \cdot (h_a^{(\ell)} \otimes d^{(\ell)}((D_H(x^\alpha))^s) t^\ell),
\end{aligned}$$

where $A(i, k) = \prod_{j=0}^i (j\alpha_k - (j-1)\alpha_{-k})$ with $A(i, k) = 1$ for $i < 0$.

PROOF. For (i), use induction on s . This is true for $s = 1$ since

$$\begin{aligned}
\text{ad } D_H(x^\alpha) \cdot e &= [D_H(x^\alpha), D_H(x^{\epsilon_k+\epsilon_m})] \\
&= \alpha_{-k} D_H(x^{\alpha+\epsilon_m-\epsilon_{-k}}) - \sigma(m) \alpha_{-m} D_H(x^{\alpha+\epsilon_k-\epsilon_{-m}}).
\end{aligned}$$

For $s \geq 1$, we have

$$\begin{aligned}
(\text{ad } D_H(x^\alpha))^{s+1} \cdot e &= \sum_{i=0}^s (-\sigma(m))^i \binom{s}{i} A(s-i-1, k) A(i-1, m) \times \\
& \quad \times [D_H(x^\alpha), D_H(x^{s\alpha-i(\epsilon_m+\epsilon_{-m})-(s-i)(\epsilon_k+\epsilon_{-k})+\epsilon_k+\epsilon_m})],
\end{aligned}$$

where

$$\begin{aligned}
&[D_H(x^\alpha), D_H(x^{s\alpha-i(\epsilon_m+\epsilon_{-m})-(s-i)(\epsilon_k+\epsilon_{-k})+\epsilon_k+\epsilon_m})] \\
&= ((s-i)\alpha_k - (s-i-1)\alpha_{-k}) D_H(x^{(s+1)\alpha-i(\epsilon_m+\epsilon_{-m})-(s-i)(\epsilon_k+\epsilon_{-k})-\epsilon_{-k}+\epsilon_m})
\end{aligned}$$

$$- \sigma(m) (i\alpha_m - (i-1)\alpha_{-m}) D_H(x^{(s+1)\alpha - i(\epsilon_m + \epsilon_{-m}) - (s-i)(\epsilon_k + \epsilon_{-k}) + \epsilon_k - \epsilon_{-m}}).$$

So we can get

$$\begin{aligned} & (\text{ad } D_H(x^\alpha))^{s+1} \cdot e \\ &= \sum_{i=0}^s (-\sigma(m))^i \binom{s}{i} A(s-i-1, k) A(i-1, m) \times \\ & \quad \times \left(((s-i)\alpha_k - (s-i-1)\alpha_{-k}) D_H(x^{(s+1)\alpha - i(\epsilon_m + \epsilon_{-m}) - (s-i)(\epsilon_k + \epsilon_{-k}) - \epsilon_{-k} + \epsilon_m}) \right. \\ & \quad \left. - \sigma(m) (i\alpha_m - (i-1)\alpha_{-m}) D_H(x^{(s+1)\alpha - i(\epsilon_m + \epsilon_{-m}) - (s-i)(\epsilon_k + \epsilon_{-k}) + \epsilon_k - \epsilon_{-m}}) \right) \\ &= \sum_{i=0}^s (-\sigma(m))^i \binom{s}{i} A(s-i, k) A(i-1, m) D_H(x^{(s+1)\alpha - i(\epsilon_m + \epsilon_{-m}) - (s-i)(\epsilon_k + \epsilon_{-k}) - \epsilon_{-k} + \epsilon_m}) \\ & \quad + \sum_{i=0}^s (-\sigma(m))^{i+1} \binom{s}{i} A(s-i-1, k) A(i, m) D_H(x^{(s+1)\alpha - i(\epsilon_m + \epsilon_{-m}) - (s-i)(\epsilon_k + \epsilon_{-k}) + \epsilon_k - \epsilon_{-m}}) \\ &= \sum_{i=0}^s (-\sigma(m))^i \binom{s}{i} A(s-i, k) A(i-1, m) D_H(x^{(s+1)\alpha - i(\epsilon_m + \epsilon_{-m}) - (s-i)(\epsilon_k + \epsilon_{-k}) - \epsilon_{-k} + \epsilon_m}) \\ & \quad + \sum_{i=1}^{s+1} (-\sigma(m))^i \binom{s}{i-1} A(s-i, k) A(i-1, m) D_H(x^{(s+1)\alpha - (i-1)(\epsilon_m + \epsilon_{-m}) - (s-i+1)(\epsilon_k + \epsilon_{-k}) + \epsilon_k - \epsilon_{-m}}) \\ &= \sum_{i=0}^{s+1} (-\sigma(m))^i \binom{s+1}{i} A(s-i, k) A(i-1, m) D_H(x^{(s+1)\alpha - i(\epsilon_m + \epsilon_{-m}) - (s+1-i)(\epsilon_k + \epsilon_{-k}) + \epsilon_k - \epsilon_{-m}}). \end{aligned}$$

For (ii), by Lemma 4.1, we have

$$\begin{aligned} ((D_H(x^\alpha))^s \otimes 1) \cdot F_a &= ((D_H(x^\alpha))^s \otimes 1) \cdot \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes e^r t^r \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} h_{a+s(\alpha_{-k}-\alpha_k)}^{(r)} (D_H(x^\alpha))^s \otimes e^r t^r \\ &= \left(\sum_{r=0}^{\infty} \frac{1}{r!} h_{a+s(\alpha_{-k}-\alpha_k)}^{(r)} \otimes e^r t^r \right) ((D_H(x^\alpha))^s \otimes 1) \\ &= F_{a+s(\alpha_{-k}-\alpha_k)} \cdot ((D_H(x^\alpha))^s \otimes 1). \end{aligned}$$

For (iii), using induction on s . For $s = 1$, we have

$$\begin{aligned} D_H(x^\alpha) \cdot u_a &= D_H(x^\alpha) \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-a}^{[r]} e^r t^r \right) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-a+(\alpha_{-k}-\alpha_k)}^{[r]} D_H(x^\alpha) \cdot e^r t^r \end{aligned}$$

$$\begin{aligned}
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-a+(\alpha_{-k}-\alpha_k)}^{[r]} \sum_{\ell=0}^r (-1)^\ell \ell! \binom{r}{\ell} e^{r-\ell} \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) t^r \\
&= \sum_{r,\ell=0}^{\infty} \frac{(-1)^{r+\ell}}{(r+\ell)!} h_{-a+(\alpha_{-k}-\alpha_k)}^{[r+\ell]} (-1)^\ell \ell! \binom{r+\ell}{\ell} e^r \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) t^{r+\ell} \\
&= \sum_{r,\ell=0}^{\infty} \frac{(-1)^r}{r!} h_{-a+(\alpha_{-k}-\alpha_k)}^{[r]} h_{-a+(\alpha_{-k}-\alpha_k)-r}^{[\ell]} e^r \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) t^{r+\ell} \\
&= \sum_{r,\ell=0}^{\infty} \frac{(-1)^r}{r!} h_{-a+(\alpha_{-k}-\alpha_k)}^{[r]} e^r t^r h_{-a+(\alpha_{-k}-\alpha_k)}^{[\ell]} \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) t^\ell \\
&= u_{a+(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} h_{-a+(\alpha_{-k}-\alpha_k)}^{[\ell]} \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) t^\ell \\
&= u_{a+(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha+(\ell-j)(\epsilon_k-\epsilon_{-m})+j(\epsilon_m-\epsilon_{-k})}) h_{-a+\ell}^{[\ell]} t^\ell \\
&= u_{a+(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}(D_H(x^\alpha)) h_{1-a}^{(\ell)} t^\ell.
\end{aligned}$$

Suppose $s \geq 1$,

$$\begin{aligned}
(D_H(x^\alpha))^{s+1} \cdot u_a &= D_H(x^\alpha) \cdot u_{a+s(\alpha_k-\alpha_{-k})} \sum_{r=0}^{\infty} d^{(r)}((D_H(x^\alpha))^s) h_{1-a}^{(r)} \\
&= u_{a+(s+1)(\alpha_k-\alpha_{-k})} \sum_{\ell'=0}^{\infty} d^{(\ell')}(D_H(x^\alpha)) h_{1-a-s(\alpha_k-\alpha_{-k})}^{(\ell')} t^{\ell'} \cdot \sum_{\ell=0}^{\infty} d^{(\ell)}((D_H(x^\alpha))^s) h_{1-a}^{(\ell)} t^\ell \\
&= u_{a+(s+1)(\alpha_k-\alpha_{-k})} \sum_{\ell',\ell=0}^{\infty} d^{(\ell')}(D_H(x^\alpha)) d^{(\ell)}((D_H(x^\alpha))^s) h_{1-a+\ell}^{(\ell')} h_{1-a}^{(\ell)} t^{\ell+\ell'} \\
&= u_{a+(s+1)(\alpha_k-\alpha_{-k})} \sum_{\ell+\ell'=0}^{\infty} d^{(\ell+\ell')}((D_H(x^\alpha))^{s+1}) h_{1-a}^{(\ell+\ell')} t^{\ell+\ell'} \\
&= u_{a+(s+1)(\alpha_k-\alpha_{-k})} \sum_{\ell=0}^{\infty} d^{(\ell)}((D_H(x^\alpha))^{s+1}) h_{1-a}^{(\ell)} t^\ell.
\end{aligned}$$

For (iv), using induction on s . For $s = 1$, we have

$$\begin{aligned}
(1 \otimes D_H(x^\alpha)) \cdot F_a &= (1 \otimes D_H(x^\alpha)) \cdot \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes e^r t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes \sum_{\ell=0}^r (-1)^\ell \ell! \binom{r}{\ell} e^{r-\ell} d^{(\ell)}(D_H(x^\alpha)) t^r \\
&= \sum_{r,\ell=0}^{\infty} \frac{\ell!}{(r+\ell)!} h_a^{(r+\ell)} \otimes (-1)^\ell \binom{r+\ell}{\ell} e^r d^{(\ell)}(D_H(x^\alpha)) t^{r+\ell}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{r,\ell=0}^{\infty} \frac{1}{r!} h_{a+\ell}^{(r)} h_a^{(\ell)} \otimes (-1)^\ell e^r d^{(\ell)}(D_H(x^\alpha)) t^{r+\ell} \\
&= \sum_{\ell=0}^{\infty} (-1)^\ell \left(\sum_{r=0}^{\infty} \frac{1}{r!} h_{a+\ell}^{(r)} \otimes e^r t^r \right) (h_a^{(\ell)} \otimes d^{(\ell)}(D_H(x^\alpha)) t^\ell) \\
&= \sum_{r=0}^{\infty} (-1)^\ell F_{a+\ell} (h_a^{(\ell)} \otimes d^{(\ell)}(D_H(x^\alpha))) t^\ell.
\end{aligned}$$

Suppose $s \geq 1$,

$$\begin{aligned}
(1 \otimes (D_H(x^\alpha))^{s+1}) &= (1 \otimes D_H(x^\alpha)) \left(\sum_{\ell=0}^{\infty} (-1)^\ell F_{a+\ell} h_a^{(\ell)} \otimes d^{(\ell)}(D_H(x^\alpha)) t^\ell \right) \\
&= \sum_{\ell,\ell'=0}^{\infty} (-1)^{\ell'} F_{a+\ell+\ell'} (h_{a+\ell}^{(\ell')} \otimes d^{(\ell')}(D_H(x^\alpha)) t^{\ell'}) \left((-1)^\ell (h_a^{(\ell)} \otimes d^{(\ell)}(D_H(x^\alpha)) t^\ell) \right) \\
&= \sum_{\ell,\ell'=0}^{\infty} (-1)^{\ell+\ell'} F_{a+\ell+\ell'} h_{a+\ell}^{(\ell')} h_a^{(\ell)} \otimes d^{(\ell')}(D_H(x^\alpha)) d^{(\ell)}(D_H(x^\alpha)) t^{\ell+\ell'} \\
&= \sum_{\ell+\ell'=0}^{\infty} (-1)^{\ell+\ell'} F_{a+\ell+\ell'} h_a^{\ell+\ell'} h_a^{\ell+\ell'} \otimes d^{(\ell+\ell')}(D_H(x^\alpha))^{s+1} t^{\ell+\ell'} \\
&= \sum_{\ell=0}^{\infty} (-1)^\ell F_{a+\ell} h_a^{(\ell)} \otimes d^{(\ell)}(D_H(x^\alpha))^{s+1} t^\ell.
\end{aligned}$$

This completes the proof. \square

LEMMA 4.3. *Fix the given two distinguished elements $h := D_H(x^{\epsilon_k + \epsilon_{-k}})$ and $e := D_H(x^{\epsilon_k + \epsilon_m})$, ($1 \leq k, |m| \leq n, m \neq \pm k$), the corresponding horizontal quantization of $U(\mathbf{H}_{\mathbb{Z}}^+)$ over $U(\mathbf{H}_{\mathbb{Z}}^+)[[t]]$ by Drinfel'd twist $\mathcal{F}(k, m)$ with the product undeformed is given by*

$$\begin{aligned}
(4.1) \quad \Delta(D_H(x^\alpha)) &= D_H(x^\alpha) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{\infty} (-1)^\ell h^{(\ell)} \otimes \\
&\quad (1-et)^{-\ell} \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k})}),
\end{aligned}$$

$$\begin{aligned}
(4.2) \quad S(D_H(x^\alpha)) &= -(1-et)^{\alpha_{-k} - \alpha_k} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} (-1)^\ell A_j B_{\ell-j} \times \\
&\quad \times D_H(x^{\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k})}) h_1^{(\ell)} t^\ell,
\end{aligned}$$

$$(4.3) \quad \varepsilon(D_H(x^\alpha)) = 0.$$

For later use, we need to make the following

LEMMA 4.4. *For $s \geq 1$, one has:*

$$\Delta((D_H(x^\alpha))^s) = \sum_{\substack{0 \leq j \leq s \\ \ell \geq 0}} \binom{s}{j} (-1)^\ell (D_H(x^\alpha))^j h^{(\ell)} \otimes (1-et)^{j(\alpha_k - \alpha_{-k}) - \ell} d^{(\ell)}((D_H(x^\alpha))^{s-j}) t^\ell.$$

$$S((D_H(x^\alpha))^s) = (-1)^s (1-et)^{-s(\alpha_k - \alpha_{-k})} \cdot \left(\sum_{\ell=0}^{\infty} d^{(\ell)}((D_H(x^\alpha))^s) \cdot h_1^{(\ell)} t^\ell \right).$$

We firstly make *the modulo p reduction* for the quantizations of $U(\mathbf{H}_{\mathbb{Z}}^+)$ in Lemma 4.3 to yield the horizontal quantizations of $U(\mathbf{H}(2n; \underline{1}))$ over $U_t(\mathbf{H}(2n; \underline{1}))$.

THEOREM 4.5. *For the given two distinguished elements $h = D_H(x^{(\epsilon_k + \epsilon_{-k})})$, $e = D_H(x^{(\epsilon_k + \epsilon_m)})$ ($1 \leq |m| \neq k \leq n$), the corresponding horizontal quantization of $U(\mathbf{H}(2n; \underline{1}))$ over $U_t(\mathbf{H}(2n; \underline{1}))$ with the product undeformed is given by*

$$(4.4) \quad \Delta(D_H(x^{(\alpha)})) = D_H(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_{-k}} \\ + \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} D_H(x^{(\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k}))}),$$

(4.5)

$$S(D_H(x^{(\alpha)})) = -(1-et)^{\alpha_k - \alpha_{-k}} \sum_{\ell=0}^{p-1} (-1)^\ell \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} D_H(x^{(\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k}))})$$

(4.6)

$$\varepsilon(D_H(x^{(\alpha)})) = 0,$$

where $0 < \alpha < \tau$, $\bar{A}_j \equiv \binom{\alpha_m + j}{j} \pmod{p}$, for $0 \leq j \leq \alpha_{-k}$, $\bar{B}_{\ell-j} \equiv \sigma(m)^{\ell-j} \binom{\alpha_k + \ell - j}{j} \pmod{p}$, for $0 \leq \ell - j \leq \alpha_{-m}$, and otherwise, $\bar{A}_j = \bar{B}_{\ell-j} = 0$.

PROOF. Note that the elements $\frac{1}{\alpha!} D_H(x^\alpha)$ in $\mathbf{H}_{\mathcal{K}}^+$ for $0 < \alpha < \tau$ will be identified with $D_H(x^{(\alpha)})$ in $\mathbf{H}(2n; \underline{1})$ and those in $J_{\underline{1}}$ (given in Section 3.2) with 0. Hence, by Lemma 4.3, we get

$$\Delta(D_H(x^{(\alpha)})) = \frac{1}{\alpha!} \Delta(D_H(x^\alpha)) = D_H(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_{-k}} \\ + \sum_{\ell=0}^{p-1} \frac{1}{\alpha!} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} \sum_{j=0}^{\ell} A_j B_{\ell-j} D_H(x^{\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k})}) \\ = D_H(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{p-1} \sum_{j=0}^{\ell} \frac{(\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k}))!}{\alpha!} \\ A_j B_{\ell-j} h^{(\ell)} \otimes (1-et)^{-\ell} D_H(x^{\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k})})$$

We can easily see that:

$$\frac{(\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k}))!}{\alpha!} A_j B_{\ell-j} \\ = A_j \frac{(\alpha_{-k} - j)! (\alpha_m + j)!}{\alpha_{-k}! \alpha_m!} B_{\ell-j} \frac{(\alpha_{-m} - (\ell-j))! (\alpha_k + (\ell-j))!}{\alpha_k! \alpha_{-m}!} \\ = \bar{A}_j \bar{B}_{\ell-j}$$

Therefore, we verify first formula.

Applying a similar argument to the antipode, we can get the other formulas.

This completes the proof. \square

To describe $\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1}))$ explicitly, we still need an auxiliary lemma.

LEMMA 4.6. *Set $e = D_H(x^{\epsilon_k + \epsilon_m})$, $d^{(\ell)} = \frac{1}{\ell!} \text{ad } e$. Then:*

(i) $d^{(\ell)}(D_H(x^{(\alpha)})) = \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} D_H(x^{(\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k}))})$; where \bar{A}_j and \bar{B}_j as in Theorem 4.5;

(ii) $d^{(\ell)}(D_H(x^{\epsilon_i + \epsilon_{-i}})) = \delta_{\ell,0} D_H(x^{\epsilon_i + \epsilon_{-i}}) + \delta_{\ell,1} (\delta_{i,-m} - \delta_{i,m} - \delta_{i,k}) e$;

(iii) $d^{(\ell)}((D_H(x^{(\alpha)}))^p) = \delta_{\ell,0} (D_H(x^{(\alpha)}))^p - \delta_{\ell,1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} - \sigma(m) \delta_{\alpha, \epsilon_m + \epsilon_{-m}}) e$.

PROOF. For (i), by (iii) of Lemma 4.1 and the proof of 4.5, we have

$$\begin{aligned} d^{(\ell)}(D_H(x^{(\alpha)})) &= \frac{1}{\alpha!} d^{(\ell)}(D_H(x^\alpha)) \\ &= \sum_{j=0}^{\ell} \frac{(\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k}))!}{\alpha!} A_j B_{\ell-j} \times \\ &\quad \times D_H(x^{(\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k}))}) \\ &= \sum_{j=0}^{\ell} \bar{A}_j \bar{B}_{\ell-j} D_H(x^{(\alpha + (\ell-j)(\epsilon_k - \epsilon_{-m}) + j(\epsilon_m - \epsilon_{-k}))}). \end{aligned}$$

For (ii), $d(D_H(x^{\epsilon_i + \epsilon_{-i}})) = [D_H(x^{\epsilon_k + \epsilon_m}), D_H(x^{\epsilon_i + \epsilon_{-i}})] = (\delta_{i,-m} - \delta_{i,m} - \delta_{i,k}) e$.

So when $\ell \geq 2$, $d^{(\ell)}(D_H(x^{\epsilon_i + \epsilon_{-i}})) = 0$. Thus, we can get (ii).

For (iii), for $\ell = 1$, we have

$$\begin{aligned} d((D_H(x^{(\alpha)}))^p) &= [e, (D_H(x^{(\alpha)}))^p] \\ &= \sum_{\ell=1}^p (-1)^\ell \binom{p}{\ell} (D_H(x^{(\alpha)}))^{p-\ell} \cdot (\text{ad } D_H(x^{(\alpha)}))^\ell(e) \\ &\equiv (-1)^p (\text{ad } D_H(x^{(\alpha)}))^p(e) \pmod{p} \\ &= -\frac{1}{(\alpha!)^p} (\text{ad } D_H(x^\alpha))^p \cdot D_H(x^{\epsilon_k + \epsilon_m}) \\ &= -\frac{1}{(\alpha!)^p} \sum_{i=0}^p \binom{p}{i} (-\sigma(m))^i A(p-i-1, k) A(i-1, m) \times \\ &\quad \times D_H(x^{p\alpha - i(\epsilon_m + \epsilon_{-m}) - (s-i)(\epsilon_k + \epsilon_{-k}) + \epsilon_k + \epsilon_m}) \\ &= -\frac{1}{(\alpha!)^p} A(p-1, k) D_H(x^{p(\alpha - \epsilon_k - \epsilon_{-k}) + \epsilon_m + \epsilon_k}) \\ &\quad + \frac{1}{(\alpha!)^p} \sigma(m)^p A(p-1, m) D_H(x^{p(\alpha - \epsilon_m - \epsilon_{-m}) + \epsilon_m + \epsilon_k}) \\ &= -\frac{1}{(\alpha!)^p} \delta_{\alpha, \epsilon_k + \epsilon_{-k}} D_H(x^{\epsilon_k + \epsilon_m}) + \frac{1}{(\alpha!)^p} \delta_{\alpha, \epsilon_m + \epsilon_{-m}} \sigma(m)^p D_H(x^{\epsilon_k + \epsilon_m}) \pmod{J} \\ &= -(\delta_{\alpha, \epsilon_k + \epsilon_{-k}} - \sigma(m) \delta_{\alpha, \epsilon_m + \epsilon_{-m}}) e. \end{aligned}$$

So when $\ell \geq 2$, $d^{(\ell)}(D_H(x^{(\alpha)}))^p = 0$. Thus, we can get (iii). \square

Based on Theorem 4.5 and Lemma 4.1, we arrive at:

THEOREM 4.7. *For the given two distinguished elements $h := D_H(x^{(\epsilon_k + \epsilon_{-k})})$, $e := D_H(x^{(\epsilon_k + \epsilon_m)})$, with $1 \leq k \neq |m| \leq n$; there exists a noncommutative and noncocommutative Hopf algebra (of horizontal type) $(\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})), m, \iota, \Delta, S, \varepsilon)$ over $\mathcal{K}[t]_p^{(q)}$ with the product undeformed, whose coalgebra structure is given by*

(4.7)

$$\Delta(D_H(x^{(\alpha)})) = D_H(x^{(\alpha)}) \otimes (1-et)^{\alpha_k - \alpha_{-k}} + \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}(D_H(x^{(\alpha)})) t^\ell,$$

$$(4.8) \quad S(D_H(x^{(\alpha)})) = -(1-et)^{\alpha_k - \alpha_{-k}} \sum_{\ell=0}^{p-1} d^{(\ell)}(D_H(x^{(\alpha)})) h_1^{(\ell)} t^\ell,$$

(4.9)

$$\varepsilon(D_H(x^{(\alpha)})) = 0,$$

where $0 < \alpha < \tau$, which is finite dimensional with $\dim_{\mathcal{K}} \mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})) = p^{2n-1}$.

PROOF. Utilizing the same arguments as in the proofs of Theorems 3.7, we shall show that the ideal $I_{t,q}$ is a Hopf ideal of the *twisted* Hopf algebra $U_{t,q}(\mathbf{H}(2n; \underline{1}))$ as in Theorem 3.3. To this end, it suffices to verify that Δ and S preserve the generators in $I_{t,q}$.

(I) By Lemma 4.4, Theorem 4.5, and Lemma 4.1, we obtain

$$\begin{aligned} & \Delta((D_H(x^{(\alpha)}))^p) \\ &= \sum_{\substack{0 \leq j \leq p \\ \ell \geq 0}} \binom{p}{j} (-1)^\ell (D_H(x^{(\alpha)}))^j h^{(\ell)} \otimes (1-et)^{j(\alpha_k - \alpha_{-k}) - \ell} d^{(\ell)}((D_H(x^{(\alpha)}))^{p-j}) t^\ell \\ &= \sum_{\ell=0}^{p-1} (-1)^\ell h^{(\ell)} \otimes (1-et)^{-\ell} d^{(\ell)}(D_H(x^{(\alpha)}))^p t^\ell \\ & \quad + \sum_{\ell=0}^{p-1} (-1)^\ell (D_H(x^{(\alpha)}))^p h^{(\ell)} \otimes (1-et)^{p(\alpha_k - \alpha_{-k}) - \ell} t^\ell \pmod{p} \\ &= 1 \otimes (D_H(x^{(\alpha)}))^p + (D_H(x^{(\alpha)}))^p \otimes 1 + h \otimes (1-et)^{-1} (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} - \sigma(m) \delta_{\alpha, \epsilon_m + \epsilon_{-m}}) et. \end{aligned}$$

That is,

$$(4.10) \quad \begin{aligned} \Delta((D_H(x^{(\alpha)}))^p) &= 1 \otimes (D_H(x^{(\alpha)}))^p + (D_H(x^{(\alpha)}))^p \otimes 1 \\ & \quad + h \otimes (1-et)^{-1} \delta_{\alpha, \epsilon_i + \epsilon_{-i}} (\delta_{k,i} + \delta_{m,i} - \delta_{-m,i}) et. \end{aligned}$$

So, if $\alpha \neq \epsilon_i + \epsilon_{-i}$, $1 \leq i \leq n$, we get:

$$\begin{aligned} \Delta((D_H(x^{(\alpha)}))^p) &= (D_H(x^{(\alpha)}))^p \otimes 1 + 1 \otimes (D_H(x^{(\alpha)}))^p \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{H}(2n; \underline{1})) + U_{t,q}(\mathbf{H}(2n; \underline{1})) \otimes I_{t,q}; \end{aligned}$$

and when $\alpha = \epsilon_i + \epsilon_{-i}$, by Theorem 4.5, we have:

$$\begin{aligned}\Delta(D_H(x^{(\epsilon_i + \epsilon_{-i})})) &= 1 \otimes D_H(x^{(\epsilon_i + \epsilon_{-i})}) + D_H(x^{(\epsilon_i + \epsilon_{-i})}) \otimes 1 \\ &\quad - h \otimes (1 - et)^{-1}(\delta_{-m,i} - \delta_{m,i} - \delta_{k,i})et \\ &= 1 \otimes D_H(x^{(\epsilon_i + \epsilon_{-i})}) + D_H(x^{(\epsilon_i + \epsilon_{-i})}) \otimes 1 \\ &\quad + h \otimes (1 - et)^{-1}(\delta_{k,i} + \delta_{m,i} - \delta_{-m,i})et.\end{aligned}$$

Combining this with 4.10, we obtain

$$\begin{aligned}\Delta((D_H x^{(\epsilon_i + \epsilon_{-i})})^p - D_H x^{(\epsilon_i + \epsilon_{-i})}) &\equiv ((D_H(x^{(\epsilon_i + \epsilon_{-i})}))^p - D_H(x^{(\epsilon_i + \epsilon_{-i})})) \otimes 1 \\ &\quad + 1 \otimes ((D_H(x^{(\epsilon_i + \epsilon_{-i})}))^p - D_H(x^{(\epsilon_i + \epsilon_{-i})})) \\ &\in I_{t,q} \otimes U_{t,q}(\mathbf{H}(2n; \underline{1})) + U_{t,q}(\mathbf{H}(2n; \underline{1})) \otimes I_{t,q}.\end{aligned}$$

(II) By Lemmas 4.4, 4.6 and Theorem 4.5, we have:

$$\begin{aligned}S((D_H(x^{(\alpha)}))^p) &= (-1)^p e^{p(\alpha - k - \alpha_k)} \sum_{\ell=0}^{\infty} d^{(\ell)}((D_H(x^{(\alpha)}))^p) h_1^{(\ell)} t^\ell \\ &= -(D_H(x^{(\alpha)}))^p + (\delta_{\alpha, \epsilon_k + \epsilon_{-k}} - \sigma(m) \delta_{\alpha, \epsilon_m + \epsilon_{-m}}) e h_1^{(1)} t \\ &= -(D_H(x^{(\alpha)}))^p + \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{k,i} e h_1^{(1)} t \\ &\quad + \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{m,i} e h_1^{(1)} t - \delta_{\alpha, \epsilon_i + \epsilon_{-i}} \delta_{-m,i} e h_1^{(1)} t,\end{aligned}$$

and when $\alpha \neq \epsilon_i + \epsilon_{-i}$, we have: $S((D_H(x^{(\alpha)}))^p) = -(D_H(x^{(\alpha)}))^p \in I_{t,q}$ when $\alpha = \epsilon_i + \epsilon_{-i}$, we have:

$$\begin{aligned}S(D_H(x^{(\epsilon_i + \epsilon_{-i})})) &= - \sum_{\ell=0}^{p-1} d^{(\ell)}(D_H(x^{(\alpha)})) h_1^{(\ell)} t^\ell \\ &= -(D_H(x^{(\epsilon_i + \epsilon_{-i})})) + (\delta_{-m,i} - \delta_{m,i} - \delta_{k,i}) e h_1^{(1)} t \\ &= -D_H(x^{(\epsilon_i + \epsilon_{-i})}) - (\delta_{-m,i} - \delta_{m,i} - \delta_{k,i}) e h_1^{(1)} t.\end{aligned}$$

So, we have:

$$S((D_H x^{(\epsilon_i + \epsilon_{-i})})^p - D_H x^{(\epsilon_i + \epsilon_{-i})}) = -((D_H x^{(\epsilon_i + \epsilon_{-i})})^p - D_H x^{(\epsilon_i + \epsilon_{-i})}) \in I_{t,q}.$$

Thereby, we show that $I_{t,q}$ is preserved by the antipode S of $U_{t,q}(\mathbf{H}(2n; \underline{1}))$ as in Theorem 4.5.

So, $I_{t,q}$ is a Hopf ideal in $U_{t,q}(\mathbf{H}(2n; \underline{1}))$. We get a finite-dimensional horizontal quantization on $\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1}))$. \square

4.2. Jordanian modular quantizations of $\mathbf{u}(\mathfrak{sp}_{2n})$. Let $\mathbf{u}(\mathfrak{sp}_{2n})$ denote the restricted universal enveloping algebra of \mathfrak{sp}_{2n} . Since Drinfel'd twists $\mathcal{F}(k; m)$ of horizontal type act stably on the subalgebra $U((\mathbf{H}_{\mathbb{Z}}^+)_0)[[t]]$, consequently on $\mathbf{u}_{t,q}(\mathbf{H}(2n; \underline{1})_0)$, these give rise to the Jordanian quantizations on $\mathbf{u}_{t,q}(\mathfrak{sp}_{2n})$.

By Lemma 4.6, we have: for $r \neq s$,

$$\begin{aligned}
d^{(\ell)}(D_H(x^{(\epsilon_r + \epsilon_s)})) &= \delta_{\ell,0} D_H(x^{(\epsilon_r + \epsilon_s)}) \\
&\quad + \delta_{\ell,1} (\sigma(m)(\delta_{-m,r} + \delta_{-m,s}) D_H(x^{(\epsilon_k + \epsilon_r + \epsilon_s - \epsilon_{-m})}) \\
&\quad - (\delta_{-k,r} + \delta_{-k,s}) D_H(x^{(\epsilon_m + \epsilon_r + \epsilon_s - \epsilon_{-k})})) \\
&\quad - \delta_{\ell,2} \sigma(m)(\delta_{-m,r} \delta_{-k,s} + \delta_{-m,s} \delta_{-k,r}) e; \\
d^{(\ell)}(D_H(x^{(2\epsilon_{-k})})) &= \delta_{\ell,0} D_H(x^{(2\epsilon_{-k})}) - \delta_{\ell,1} D_H(x^{(\epsilon_{-k} + \epsilon_m)}) + \delta_{\ell,2} D_H(x^{(2\epsilon_m)}); \\
d^{(\ell)}(D_H(x^{(2\epsilon_{-m})})) &= \delta_{\ell,0} D_H(x^{(2\epsilon_{-m})}) + \delta_{\ell,1} \sigma(m) D_H(x^{(\epsilon_{-m} + \epsilon_k)}) + \delta_{\ell,2} D_H(x^{(2\epsilon_k)});
\end{aligned}$$

and for $r \neq -k, -m$, $d^{(\ell)}(D_H(x^{(2\epsilon_r)})) = \delta_{\ell,0} D_H(x^{(2\epsilon_r)})$.

By Theorem 4.7, we have:

THEOREM 4.8. *For the given two distinguished elements $h = D_H(x^{(\epsilon_k + \epsilon_{-k})})$, $e = D_H(x^{(\epsilon_k + \epsilon_m)})$ ($1 \leq |m| \neq k \leq n$), the corresponding Jordanian quantization of $\mathbf{u}(\mathbf{H}(2n, \underline{1})_0) \cong \mathbf{u}(\mathfrak{sp}_{2n})$ over $\mathbf{u}_{t,q}(\mathbf{H}(2n, \underline{1})_0) \cong \mathbf{u}_{t,q}(\mathfrak{sp}_{2n})$ with the product undeformed, whose coalgebra structure is given by*

$$\begin{aligned}
\Delta(D_H(x^{(\epsilon_r + \epsilon_s)})) &= D_H(x^{(\epsilon_r + \epsilon_s)}) \otimes (1 - et)^{\delta_{r,k} + \delta_{s,k} - \delta_{r,-k} - \delta_{s,-k}} + 1 \otimes D_H(x^{(\epsilon_r + \epsilon_s)}) \\
&\quad + (-1) h \otimes (1 - et)^{-1} (\sigma(m)(\delta_{-m,r} + \delta_{-m,s}) D_H(x^{(\epsilon_k + \epsilon_r + \epsilon_s - \epsilon_{-m})}) \\
&\quad - (\delta_{-k,r} + \delta_{-k,s}) D_H(x^{(\epsilon_m + \epsilon_r + \epsilon_s - \epsilon_{-k})})) t \\
&\quad - h^{(2)} \otimes (1 - et)^{-2} \sigma(m)(\delta_{r,-m} \delta_{s,-k} + \delta_{s,-m} \delta_{r,-k}) e t^2; \\
\Delta(D_H(x^{(2\epsilon_r)})) &= D_H(x^{(2\epsilon_r)}) \otimes (1 - et)^{2\delta_{k,r} - 2\delta_{-k,r}} + 1 \otimes D_H(x^{(2\epsilon_r)}) \\
&\quad + \delta_{r,-k} (h \otimes (1 - et)^{-1} D_H(x^{(\epsilon_{-k} + \epsilon_m)})) t \\
&\quad + h^{(2)} \otimes (1 - et)^{-2} D_H(x^{(2\epsilon_m)}) t^2 \\
&\quad + \delta_{r,-m} (-\sigma(m) h \otimes (1 - et)^{-1} D_H(x^{(\epsilon_{-m} + \epsilon_k)})) t \\
&\quad + h^{(2)} \otimes (1 - et)^{-2} D_H(x^{(2\epsilon_k)}) t^2; \\
S(D_H(x^{(\epsilon_r + \epsilon_s)})) &= -(1 - et)^{\delta_{r,-k} + \delta_{s,-k} - \delta_{rk} - \delta_{sk}} (D_H(x^{(\epsilon_r + \epsilon_s)}) \\
&\quad + ((\sigma(m) \delta_{-m,r} + \delta_{-m,s}) D_H(x^{(\epsilon_k + \epsilon_r + \epsilon_s - \epsilon_{-m})}) \\
&\quad - (\delta_{-k,r} + \delta_{-k,s}) D_H(x^{(\epsilon_m + \epsilon_r + \epsilon_s - \epsilon_{-k})})) h_1^{(1)} t \\
&\quad + \sigma(m)(\delta_{r,-m} \delta_{s,-k} + \delta_{s,-m} \delta_{r,-k}) e h_1^{(2)} t^2); \\
S(D_H(x^{(2\epsilon_r)})) &= -(1 - et)^{2\delta_{r,-k} - 2\delta_{rk}} D_H(x^{(2\epsilon_r)}) \\
&\quad + \delta_{r,-k} (1 - et)^2 (D_H(x^{(\epsilon_{-k} + \epsilon_m)}) h_1^{(1)} t - D_H(x^{(2\epsilon_m)}) h_1^{(2)} t^2) \\
&\quad - \delta_{r,-m} (\sigma(m) D_H(x^{(\epsilon_{-m} + \epsilon_k)}) h_1^{(1)} t + D_H(x^{(2\epsilon_k)}) h_1^{(2)} t^2); \\
\varepsilon(D_H(x^{(\epsilon_r + \epsilon_s)})) &= \epsilon(D_H(x^{(2\epsilon_r)})) = 0.
\end{aligned}$$

REMARK 4.9. As $\mathbf{H}(2n, \underline{1})_0 \cong \mathfrak{sp}_{2n}$, which via the identification $D_H(x^{(\epsilon_r + \epsilon_s)})$ with $\sigma(s)E_{r,-s} + \sigma(r)E_{s,-r}$ for $1 \leq |r| \neq |s| \leq n$ and $D_H(x^{(2\epsilon_r)})$ with $\sigma(r)E_{r,-r}$,

we get a Jordanian quantization for \mathfrak{sp}_{2n} , which has been discussed by Kulish et al (cf. [15], [16] etc.).

COROLLARY 4.10. For the given two distinguished elements $h = E_{k,k} - E_{-k,-k}$, $e = \sigma(m)E_{k,-m} - E_{m,-k}$ ($1 \leq k \neq |m| \leq n$), the corresponding Jordanian quantization of $\mathfrak{u}(\mathfrak{sp}_{2n})$ over $\mathfrak{u}_{t,q}(\mathfrak{sp}_{2n})$ with the product undeformed, whose coalgebra structure is given by

$$\begin{aligned}
& \Delta(\sigma(s)E_{r,-s} + \sigma(r)E_{s,-r}) \\
&= (\sigma(s)E_{r,-s} + \sigma(r)E_{s,-r}) \otimes (1-et)^{\delta_{r,k} + \delta_{s,k} - \delta_{r,-k} - \delta_{s,-k}} + 1 \otimes (\sigma(s)E_{r,-s} + \sigma(r)E_{s,-r}) \\
&\quad - h \otimes (1-et)^{-1} \left(\delta_{r,-m} \sigma(m) (-E_{s,-k} + \sigma(s)E_{k,-s}) + \delta_{s,-m} \sigma(m) (-E_{r,-k} + \sigma(r)E_{k,-r}) \right. \\
&\quad \left. - \delta_{r,-k} (\sigma(m)E_{s,-m} + \sigma(s)E_{m,-s}) - \delta_{s,-k} (\sigma(m)E_{r,-m} + \sigma(r)E_{m,-r}) \right) t \\
&\quad - h^{(2)} \otimes (1-et)^{-2} \sigma(m) (\delta_{r,-m} \delta_{s,-k} + \delta_{s,-m} \delta_{r,-k}) (\sigma(m)E_{k,-m} - E_{m,-k}) t^2; \\
& \Delta(\sigma(r)E_{r,-r}) = \sigma(r)E_{r,-r} \otimes (1-et)^{2\delta_{k,r} - 2\delta_{-k,r}} + 1 \otimes \sigma(r)E_{r,-r} \\
&\quad + \delta_{r,-k} \left(\left(h \otimes (1-et)^{-1} (\sigma(m)E_{-k,-m} + E_{m,k}) \right) t + h^{(2)} \otimes (1-et)^{-2} \sigma(m)E_{m,-m} t^2 \right) \\
&\quad + \delta_{r,-m} \left(\left(\sigma(m)h \otimes (1-et)^{-1} (\sigma(m)E_{k,m} + E_{-m,-k}) \right) t - h^{(2)} \otimes (1-et)^{-2} E_{k,-k} t^2 \right); \\
& S(\sigma(s)E_{r,-s} + \sigma(r)E_{s,-r}) = -(1-et)^{\delta_{r,-k} + \delta_{s,-k} - \delta_{r,k} - \delta_{s,k}} \left((\sigma(s)E_{r,-s} + \sigma(r)E_{s,-r}) \right. \\
&\quad \left. + \left(\delta_{r,-m} \sigma(m) (-E_{s,-k} + \sigma(s)E_{k,-s}) + \delta_{s,-m} \sigma(m) (-E_{r,-k} + \sigma(r)E_{k,-r}) \right. \right. \\
&\quad \left. \left. - \delta_{r,-k} (\sigma(m)E_{s,-m} + \sigma(s)E_{m,-s}) - \delta_{s,-k} (\sigma(m)E_{r,-m} + \sigma(r)E_{m,-r}) \right) h_1^{(1)} t \right. \\
&\quad \left. - \sigma(m) (\delta_{r,-m} \delta_{s,-k} + \delta_{s,-m} \delta_{r,-k}) (\sigma(m)E_{k,-m} - E_{m,-k}) h_1^{(2)} t^2 \right); \\
& S(\sigma(r)E_{r,-r}) = -(1-et)^{2\delta_{r,-k} - 2\delta_{r,k}} \sigma(r)E_{r,-r} \\
&\quad + \delta_{r,-k} (1-et)^2 \left((\sigma(m)E_{-k,-m} + E_{m,k}) h_1^{(1)} t - \sigma(m)E_{m,-m} h_1^{(2)} t^2 \right) \\
&\quad + \delta_{r,-m} \sigma(m) \left((\sigma(m)E_{m,k} + E_{-k,-m}) h_1^{(1)} t + E_{k,-k} h_1^{(2)} t^2 \right); \\
& \varepsilon(\sigma(s)E_{r,-s} + \sigma(r)E_{s,-r}) = \varepsilon(\sigma(r)E_{r,-r}) = 0.
\end{aligned}$$

for $1 \leq |r|, |s| \leq n$ and $r \neq s$.

EXAMPLE 4.11. For $n = 2$, take $h = E_{11} - E_{-1,-1}$, $e = E_{12} - E_{-2,-1}$, and set $h' = E_{22} - E_{-2,-2}$, $f = (1-et)^{-1}$. By Corollary 4.10, we get a Jordanian quantization on $\mathfrak{u}_{t,q}(\mathfrak{sp}_4)$ with the coproduct as follows:

$$\begin{aligned}
& \Delta(h) = 1 \otimes h + h \otimes (1-f); \\
& \Delta(h') = h' \otimes 1 + 1 \otimes h' - h \otimes f; \\
& \Delta(e) = 1 \otimes e + e \otimes f^{-1} - h \otimes f; \\
& \Delta(E_{1,-2} + E_{2,-1}) = (E_{1,-2} + E_{2,-1}) \otimes f^{-1} + 1 \otimes (E_{1,-2} + E_{2,-1}) - 2h \otimes fE_{1,-1}t;
\end{aligned}$$

$$\begin{aligned}
\Delta(E_{-1,2} + E_{-2,1}) &= (E_{-1,2} + E_{-2,1}) \otimes f + 1 \otimes (E_{-1,2} + E_{-2,1}) - 2h \otimes fE_{-2,2}t; \\
\Delta(E_{-1,-2} + E_{2,1}) &= (E_{-1,-2} + E_{2,1}) \otimes f + 1 \otimes (E_{-1,-2} + E_{2,1}) \\
&\quad - h \otimes f(h - h')t - h^{(2)} \otimes f^2et^2; \\
\Delta(E_{1,-1}) &= E_{1,-1} \otimes f^{-2} + 1 \otimes E_{1,-1}; \\
\Delta(E_{2,-2}) &= E_{2,-2} \otimes 1 + 1 \otimes E_{2,-2} - h \otimes f(E_{1,-2} + E_{2,-1})t + h^{(2)} \otimes f^2E_{1,-1}t^2; \\
\Delta(E_{-1,1}) &= E_{-1,1} \otimes f^2 + 1 \otimes E_{-1,1} + h \otimes f(E_{-1,2} + E_{-2,1})t + h^{(2)} \otimes f^2E_{-2,2}t^2; \\
\Delta(E_{-2,2}) &= E_{-2,2} \otimes 1 + 1 \otimes E_{-2,2}.
\end{aligned}$$

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